Nov 15 2022

Week 11

2020 A Adv. Cal II

Green's theorem 1 Let C be a simple, closed curve enclosing D and $\vec{F}$ is a smooth v.f. in D (up to the boundary C). then

$$\oint_C M\,dx + N\,dy = \iint_D (N_x - M_y)\,dA$$ where C

is in anticlockwise way.

Or formulated as Green's theorem for simply-connected regions:

Green's theorem 2 Let D be a simply-connected region and $\vec{F}$ smooth v.f. in D. For any simple, closed curve C in D,

$$\oint_C M\,dx + N\,dy = \iint_{D_1} (N_x - M_y)\,dA$$

when C is in anticlock way and D_1 is the region enclosed by C.

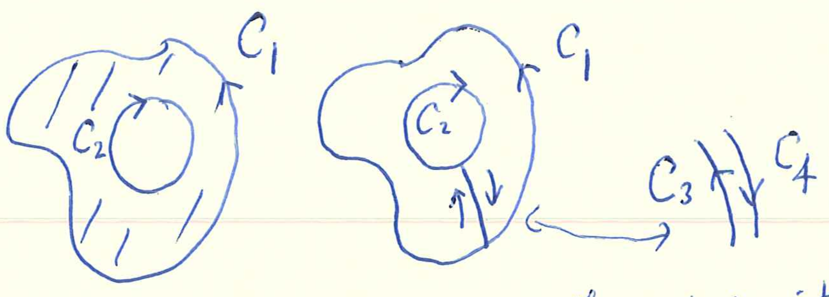
For multi-connected region we have.

Green's theorem 3 Let $C_1, C_2, \ldots C_n$ be simple, closed curve which bounds D and $\vec{F}$ a smooth v.f. in D (up to boundaries C_j's) then

$$\sum_{j=1}^{n} \oint_{C_j} M\,dx + N\,dy = \iint_D (N_x - M_y)\,dA,$$ when C_1 is (outer boundary)

anticlock, and $C_2, \ldots, C_n$ are clockwise.

The proof is by "cutting up"



The cut consisting of 2 identical arcs C_3, C_4 with opposite direction

$C = C_1 + C_3 + C_2 + C_4$

a closed curve, although not simple, but old Green's thm 1 , still holds as it encloses a region without holes.

$$\therefore \left(\oint_{C_1} + \int_{C_3} + \oint_{C_2} + \int_{C_4}\right)(M\,dx + N\,dy)$$

$$= \oint_C M\,dx + N\,dy = \iint_D (N_x - M_y)\,dA.$$

But $\int_{C_3} = -\int_{C_4}$, so $\left(\oint_{C_1} + \oint_{C_2}\right)(M\,dx + N\,dy) = \iint_D (N_x - M_y)\,dA$. #

An application:

<u>Theorem</u> For any simple, closed curve C enclosing the origin,

$$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = 2\pi, \quad C \text{ anticlockwise}.$$

Pf. Let C_2 be a small circle contained inside C. Then

our v.f is smooth in the region D bounded bet. C and C_2.

$$\left(\oint_C + \oint_{C_2}\right)\left(\frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy\right)$$

$$= \iint_D \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) dA(x,y)$$

see previous lecture

$$= \iint_D 0 \, dA = 0.$$

$$\therefore \oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$= \oint_{-C_2} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$= 2\pi$ (done before). #

Surface Integral

A <u>parametric surface</u> is a conti map : $D \to \mathbb{R}^3$ where D is a region in $\mathbb{R}^2$. Usually it is taken to be a rectangle.

Let $\vec{r}(u,v)$ be a parametric surface. It is <u>smooth</u> if $\vec{r}_u, \vec{r}_v$ are continuous and $|\vec{r}_u \times \vec{r}_v| > 0$ in the interior of D. A <u>surface</u> is a set S which is the image of a parametric surface whose interior is 1-1. It is <u>smooth</u> if the parametrization is smooth.

e.g. the sphere $x^2+y^2+z^2=a^2$, $a>0$.

We take (φ, θ) in spherical coor. as (u,v), so

$$(\varphi,\theta) \mapsto (a\sin\varphi\cos\theta,\ a\sin\varphi\sin\theta,\ a\cos\varphi) = \vec{r}(\varphi,\theta),$$

(ie $u=\varphi$, $v=\theta$), $D=[0,\pi]\times[0,2\pi]$.

$$\vec{r}_\varphi = (a\cos\varphi\cos\theta,\ a\cos\varphi\sin\theta,\ -a\sin\varphi)$$

$$\vec{r}_\theta = (-a\sin\varphi\sin\theta,\ a\sin\varphi\cos\theta,\ 0)$$

$|\vec{r}_\varphi \times \vec{r}_\theta| = a^2\sin\varphi > 0$ in $(0,\pi)\times[0,2\pi]$, ∴ this is a smooth parametrization.

e.g. the cylinder $x^2+y^2=a^2$, $a>0$.

$$(\theta, z) \mapsto (a\cos\theta,\ a\sin\theta,\ z) = \vec{r}(\theta,z)$$

$$D=[0,2\pi]\times(-\infty,\infty)$$

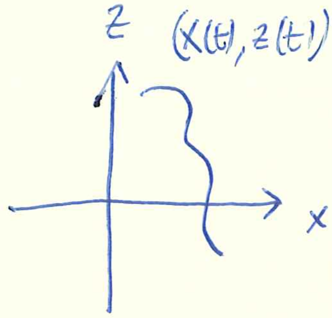
$$\vec{r}_\theta = (-a\sin\theta,\ a\cos\theta,\ 0)$$

$$\vec{r}_z = (0,\ 0,\ 1)$$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} i & j & k \\ -a\sin\theta & a\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

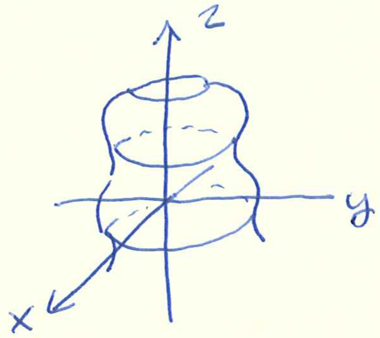
$$= a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j} + 0\,\hat{k}$$

$|\vec{r}_\theta \times \vec{r}_z| = a > 0$ $\therefore$ always a smooth parametrization.

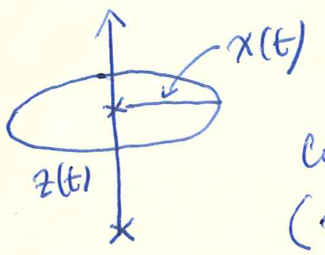


e.g. Surface of revolution

Let $t \mapsto (x(t), z(t))$ be a curve in right xz-plane. Rotate it around z-axis to obtain a surface S.



For fixed $z(t)$ (height), the cross section is a circle of radius $x(t)$. Hence, points on the circle can be described by



$(x(t)\cos\theta,\ x(t)\sin\theta,\ z(t)),\ \theta \in [0, 2\pi]$

$\therefore\ (\theta, t) \mapsto (x(t)\cos\theta,\ x(t)\sin\theta,\ z(t))$

is a parametrization of S.

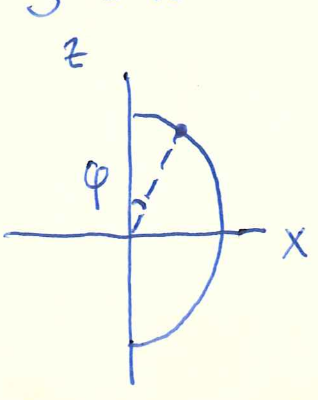
$$\vec{r}_\theta = (-x\sin\theta,\ x\cos\theta,\ 0)$$

$$\vec{r}_t = (x'\cos\theta,\ x'\sin\theta,\ z')$$

$|\vec{r}_\theta \times \vec{r}_t| = x(t)\sqrt{x'(t)^2 + y'^2(t)} > 0$ as $x(t) > 0$.

$\therefore$ S is a smooth surface.

Note

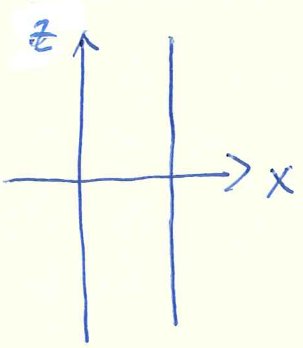


$t \leftrightarrow \varphi$

$(x(t), z(t)) \leftrightarrow (a\sin\varphi, a\cos\varphi)$

recover the sphere

$(a\sin\varphi\cos\theta,\ a\sin\varphi\sin\theta,\ a\cos\varphi)$



$(x(t), z(t)) \longleftrightarrow (a, z)$ use z as t

recover the cylinder

$(a\cos\theta, a\sin\theta, z)$.

e.g. In general, when S is the graph of $z = f(x,y)$ over $(x,y) \in D$. A natural parametrization is

$$(x, y) \longmapsto (x, y, f(x,y))$$

$$\vec{r}_x = (1, 0, f_x), \ \vec{r}_y = (0, 1, f_y)$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + f_x^2 + f_y^2} > 0 \text{ always smooth surface.}$$

e.g. Give 2 parametrizations for the cap:

$$x^2 + y^2 + z^2 = 4, \quad z \geq 1$$

① $x^2+y^2+z^2 = 4$, $z = 1$ intersect at

$$x^2 + y^2 + 1^2 = 4, \quad x^2 + y^2 = 3$$

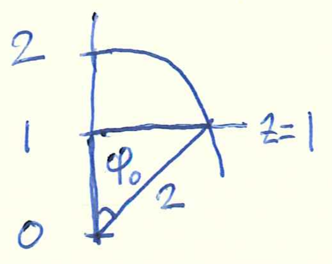
$\therefore$ S is a graph over $x^2+y^2 \leq 3$

$$(x, y) \longmapsto (x, y, \sqrt{4 - x^2 - y^2}), \ (x,y) \in D_{\sqrt{3}} \text{ (disk of radius } \sqrt{3})$$

is a smooth parametrization for the cap.

② Use spherical coor.

$(\varphi, \theta) \longmapsto (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi)$ is another smooth parametrization, the range of φ is $\therefore D = [0, \frac{\pi}{3}] \times [0, 2\pi]$.



$$\cos\varphi_0 = \frac{1}{2}$$

$$\varphi_0 = \pi/3$$

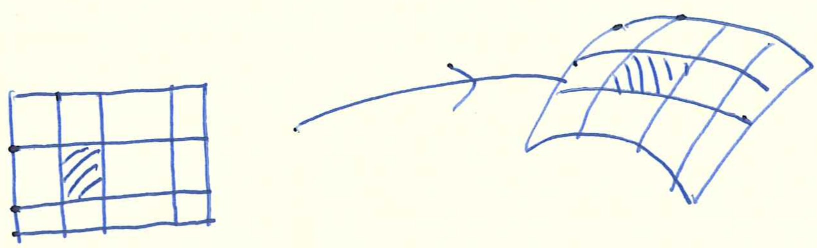
Surface Integral

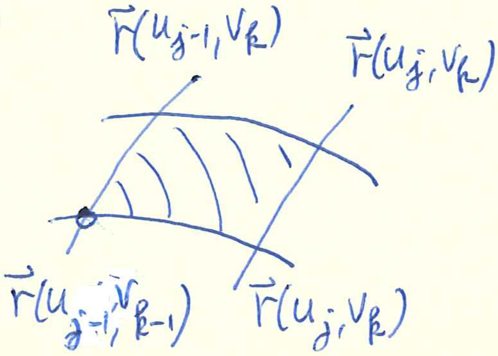
Want to define $\iint_S f\, d\sigma$ so that it is

- the mass of S with density f
- the surface area of S with $f \equiv 1$.

Again, let

$$\vec{r}(u,v) : [a,b]\times[c,d] \longrightarrow \mathbb{R}^3$$



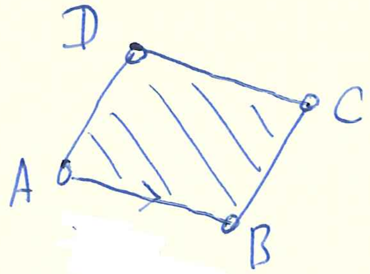


approxi mass

$$\simeq \sum f(\vec{r}(u_{j-1}, v_{k-1})) \times \text{area } ABCD$$

$$= \sum f(\vec{r}(u_{j-1}, v_{k-1})) \left| \vec{r}_u \times \vec{r}_v (u_{j-1}, v_{k-1}) \right| \Delta u_j \Delta v_k$$

$$\to \iint_{[a,b]\times[c,d]} f(\vec{r}(u,v)) \left| \vec{r}_u \times \vec{r}_v \right| (u,v)\, dA(u,v).$$



approx. by

A $\vec{r}(u_{j-1}, v_{k-1})$

B $\vec{r}(u_{j-1}, v_{k-1}) + \vec{r}_u (u_{j-1}, v_{k-1}) \Delta u_j$

C $\vec{r}(u_{j-1}, v_{k-1}) + \vec{r}_u \Delta u_j + \vec{r}_v \Delta v_k$

D $\vec{r}(u_{j-1}, v_{k-1}) + \vec{r}_v \Delta v_k$

area of $/\!/ ABCD$

$$= \left| \vec{r}_u \times \vec{r}_v (u_{j-1}, v_{k-1}) \right| \Delta u_j \Delta v_k$$

Define

$$\iint_S f\,d\sigma = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|(u,v)\,dA(u,v)$$

where $\vec{r}(u,v)$ is a smooth parametrization of S which is 1-1 in the interior. When $f \equiv 1$, this gives the surface area of S.

(Cont'd)

Examples

Remember this:

(I) graph $(x,y) \mapsto (x, y, f(x,y))$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + f_x^2 + f_y^2}$$

(II) sphere $(\varphi, \theta) \mapsto (a \sin\varphi \cos\theta, a \sin\varphi \sin\theta, a \cos\varphi)$

(a radius)

$$|\vec{r}_\varphi \times \vec{r}_\theta| = a^2 \sin\varphi$$

(III) surface of revolution $(\alpha, t) \mapsto (x(t)\cos\alpha, x(t)\sin\alpha, z(t))$

$$|\vec{r}_\alpha \times \vec{r}_t| = x(t)\sqrt{x'^2(t) + z'^2(t)}.$$

e.g. 1 Find the surface area of

(a) the sphere $x^2 + y^2 + z^2 = 16$

(b) the cap $x^2 + y^2 + z^2 = 16$ above $z = 1$.

(a) surface area of sphere: By (II),

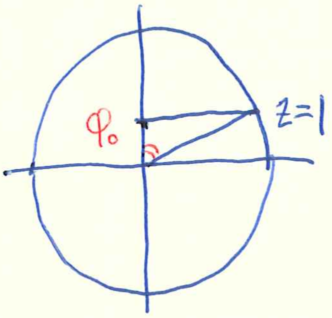
$$(\varphi, \theta) \in [0, \pi] \times [0, 2\pi]$$

$$A = \int_0^{2\pi}\int_0^{\pi} 1 \times 16 \sin\varphi \, d\varphi \, d\theta = 16 \times 2\pi \times (-\cos\varphi)\Big|_0^{\pi}$$

$$= 64\pi.$$

(b) surface area of the cap

$$\cos\varphi_0 = \frac{1}{4}, \quad \varphi_0 = \cos^{-1}\frac{1}{4}$$



$$\text{area} = \int_0^{2\pi}\int_0^{\varphi_0} 16\sin\varphi \, d\varphi \, d\theta$$

$$= 16\times 2\pi\,(-\cos\varphi)\Big|_0^{\varphi_0}$$

$$= 24\pi.$$

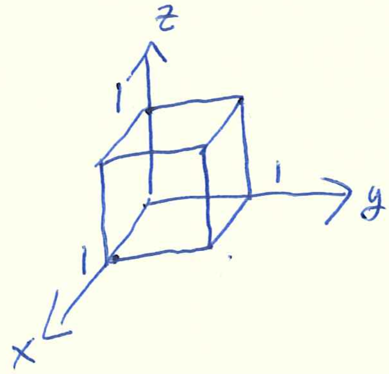
e.g. 2 Find the surface area of the portion of the plane $2x+y+z=1$ inside the cylinder $x^2+y^2=1$.

the surface $z=1-2x-y$ is a graph over $D_1: x^2+y^2\le 1$.

By (I), $d\sigma = \sqrt{1+(-2)^2+(-1)^2}\, dA = \sqrt{6}\, dA$

$$\therefore \text{area} = \iint_{D_1} 1\times\sqrt{6}\, dA = \sqrt{6}\,|D_1| = \sqrt{6}\,\pi. \ \#$$

e.g. 3 Evaluate $\iint_C xyz\, d\sigma$ when C is the unit cube $[0,1]\times[0,1]\times[0,1]$.



C consists of 6 faces. But the 3 faces sitting on xy-, yz-, zx-plane satisfying $z=0$, $x=0$, $y=0$ respectively, hence

$$f(x,y,z) = xyz = 0,$$

so they don't have contribution to the integral. It suffices to find the integrals over the rest 3 faces, ie, the faces at $x=1$, parallel to the y-z-plane, at $y=1$ parallel to the xz-plane and at $z=1$ parallel to the xy-plane. By symmetry, they all equal.

Let C_1 be the face at $z=1$, parallel t the xy-plane.

It is the graph $z \equiv 1$ over the unit square $[0,1]\times[0,1]$

~~on~~ in the xy-plane.

$$(x,y) \longmapsto (x,y,1)$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1+0+0} = 1$$

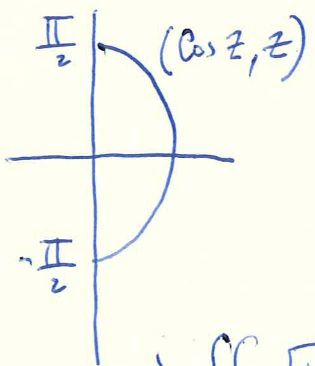
$$\therefore \iint_{C_1} xyz\, d\sigma = \iint_{[0,1]\times[0,1]} xy\,1\, dA(x,y)$$

$$= \int_0^1\int_0^1 xy\, dy\, dx = \frac{1}{4}.$$

$$\therefore \iint_C xyz\, d\sigma = 3/4 \ . \#$$

e.g. 4 Evaluate $\iint_S \sqrt{1-x^2-y^2}\, d\sigma$ where S is the surface obtained by rotating $x = \cos z$ around the z-axis.

The curve $z \mapsto (\cos z, z)$ describes $t \mapsto (x(t), z(t))$



By (III), S is described by

$$(\alpha, z) \mapsto (\cos z \cos\alpha, \cos z \sin\alpha, z)$$

$$|\vec{r}_\alpha \times \vec{r}_t| = \cos z \sqrt{1+\sin^2 z}$$

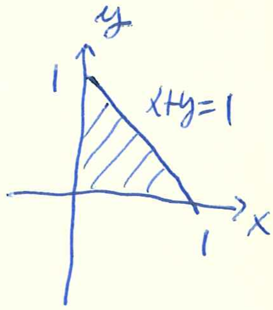
$$\therefore \iint_S \sqrt{1-x^2-y^2}\, d\sigma = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \sqrt{1-(\cos z\cos\alpha)^2-(\cos z\sin\alpha)^2}\, \cos z\sqrt{1+\sin^2 z}\, dz\, d\alpha$$

$$= \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} |\sin z| \cos z \sqrt{1+\sin^2 z}\, dz\, d\alpha$$

$$\vdots$$

$$= \frac{4\pi}{3}(2\sqrt{2}-1).$$

eg 5. Evaluate $\iint_S \sqrt{x(1+2z)}\, d\sigma$ where S is the portion of the cylinder $z = y^2/2$ over the triangle T :



This is the case of a graph over T in the xy-plane.

$$f(x,y) = y^2/2$$

$$d\sigma = \sqrt{1+f_x^2+f_y^2} = \sqrt{1+y^2}$$

$$\therefore \iint_S \sqrt{x}\sqrt{1+2z}\, d\sigma = \iint_T \sqrt{x}\sqrt{1+y^2}\sqrt{1+y^2}\, dA(x,y)$$

$$= \int_0^1 \int_0^{1-x} \sqrt{x}\,(1+y^2)\, dy\, dx$$

$$\vdots$$

$$= \frac{284}{945}. \#$$